Quarterly Report 5 | *Covering the Period 15 January to 15 April 1967*

APPLICATION OF OPTIMUM ESTIMATION AND CONTROL THEORY TO OPTICAL TRACKING PROBLEMS

NOV 1983
RECEIVED
NASA STI FACILITY
ACCESS DEPT.

By: R. M. DRESSLER

Prepared for:

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
ELECTRONICS RESEARCH CENTER
CAMBRIDGE, MASSACHUSETTS

CONTRACT NAS12-59

(NASA-CR-174557) APPLICATION OF OPTIMUM ESTIMATION AND CONTROL THEORY TO OPTICAL TRACKING PROBLEMS Quarterly Report, 15 Jan. - 15 Apr. 1967 (Stanford Research Inst.) 49 p

N84-70449

Unclas
00/74 44383

STANFORD RESEARCH INSTITUTE

MENLO PARK, CALIFORNIA

STANFORD RESEARCH INSTITUTE
MENLO PARK, CALIFORNIA

May 1967

Quarterly Report 5 | *Covering the Period 15 January to 15 April 1967*

APPLICATION OF OPTIMUM ESTIMATION AND CONTROL THEORY TO OPTICAL TRACKING PROBLEMS

By: R. M. DRESSLER

Prepared for:

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
ELECTRONICS RESEARCH CENTER
CAMBRIDGE, MASSACHUSETTS

CONTRACT NAS12-59

SRI Project 5578

Approved: PHILIP E. MERRITT, MANAGER
INFORMATION AND CONTROL LABORATORY

Copy No. 1

ABSTRACT

The work performed from 15 January to 15 April 1967 on Contract NAS12-59 is described in detail.

The report deals with the application of optimum linear estimation and control theory in designing an optical tracking system for use in deep-space communication. The estimator is obtained by using the extended Kalman filter, which necessitates the linearization of the equations of motion for the spacecraft and of the measurement equations. The controller is obtained by making the appropriate linearizations and applying new results in the theory of linear optimal control.

This study has resulted in the development of a digital computer program which simulates the operation of the estimator-controller in the optical tracking system.

CONTENTS

ABSTRACT. ii

LIST OF ILLUSTRATIONS iv

I INTRODUCTION 1

II PROBLEM FORMULATION. 3

A. Spacecraft. 3

B. Telescope Dynamics. 7

C. Measurement System. 14

D. Estimator and Controller. 20

III ESTIMATION EQUATIONS 21

A. Prediction. 22

B. Correction. 23

IV CONTROL EQUATIONS. 28

A. Steady-State Approximation. 37

V SUMMARY AND FUTURE WORK. 42

REFERENCES. 43

DD Form 1473

ILLUSTRATIONS

Fig. 1	Structure of Earth-Based Terminal.	3
Fig. 2	Relative Motion of Spacecraft and Tracking Site. .	4
Fig. 3	Mechanical Structure of Telescope.	7
Fig. 4	Tracking Configuration (Declination Channel) . . .	17

I INTRODUCTION

The objective of this project is to determine the capabilities and performance of a deep-space optical communication system and, in particular, to develop techniques for achieving the high-precision tracking required. Preliminary studies on optical communication systems are described in Technical Memoranda 8[1]* and 9.[2] A cooperative system configuration (as described in Ref. 1) will be considered, since it presently appears to be the most promising candidate for an actual system design.

The purpose of this quarterly report is to describe the application of optimum linear estimation and control theory in designing the earth-based terminal of an optical tracking system. The present study has resulted in the development of a digital-computer program which simulates the operation of the derived estimator-controller configuration in the system to track the incoming optical signal from the deep-space probe. A similar approach can be employed in synthesizing the spacecraft-borne terminal of the system.

A solution to the optical tracking problem can be obtained by solving the estimation and control portions separately. Since this problem is nonlinear, the assumption that the estimation and control portions can be solved separately may yield a solution that is not optimal in the strictest sense.[3] However, this approach is taken since, in general, the solution to the combined optimization problem is not computationally feasible.

The estimator, which generates a "best" estimate of the present state of the system (spacecraft and telescope dynamics), is derived in Sec. III. The estimation problem is solved by employing the extended Kalman filter, which necessitates the linearization of the equations of

---

*References are listed at the end of this report.

motion for the spacecraft and of the measurement equations. The estimate of the system state is then employed in the controller to compute a "best" control with respect to the given performance criterion. The control problem is solved in Sec. IV by making the appropriate linearizations and applying some new results in the theory of linear optimal control.

II PROBLEM FORMULATION

The earth-based terminal of the optical communication system being studied is shown in Fig. 1. Mathematical models for the various elements of this system are given below. The detailed derivations of these models are presented in Memorandum 9[2] and will not be repeated here.

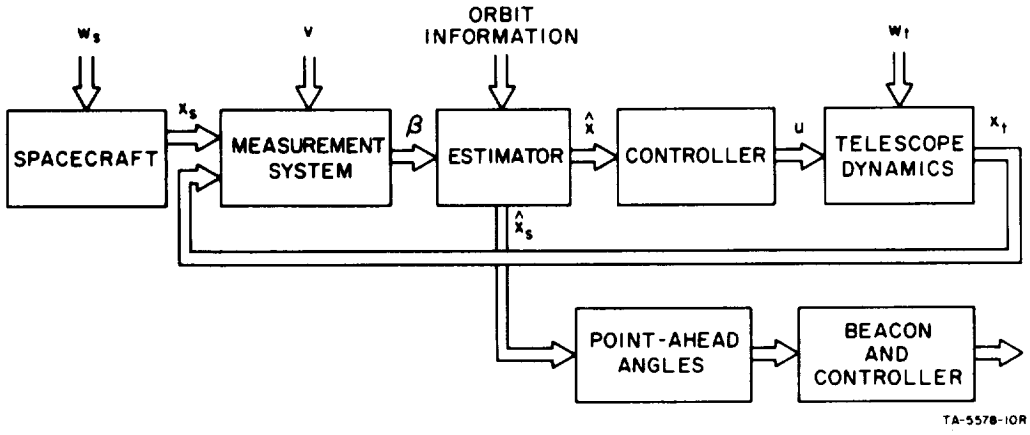

FIG. 1 STRUCTURE OF EARTH-BASED TERMINAL

The geometric angle of arrival of the incoming optical signal that is to be tracked by the earth-based receiver is determined by the relative motion of the spacecraft with respect to the tracking site. The geometry of the problem is shown in Fig. 2. The 1_s and 2_s axes, which lie in the plane of the ecliptic (the plane of the earth's orbit), and the 3_s axis, which is perpendicular to this plane, define a sun-centered Cartesian coordinate system (the 1_s axis is taken to be the winter solstice).

A. Spacecraft

A model to describe the spacecraft motion is taken to be

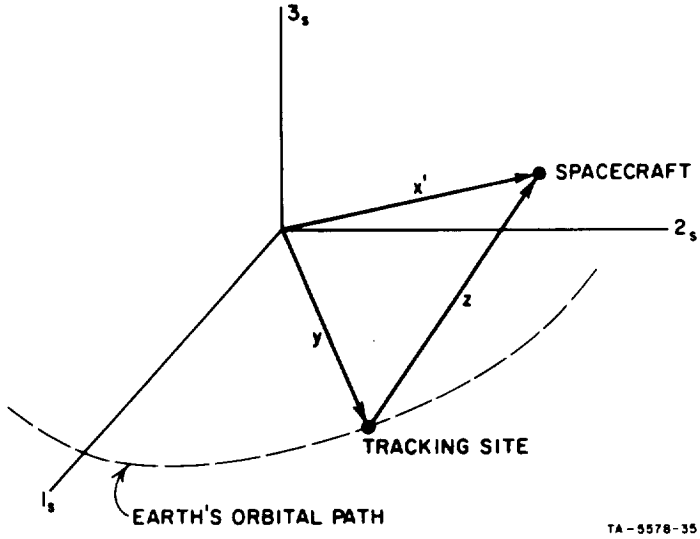

FIG. 2 RELATIVE MOTION OF SPACECRAFT AND TRACKING SITE

$$\ddot{x}_{1_s} = -\frac{\mu_s x_{1_s}}{r_s^3}$$

$$\ddot{x}_{2_s} = -\frac{\mu_s x_{2_s}}{r_s^3}$$

$$\ddot{x}_{3_s} = -\frac{\mu_s x_{3_s}}{r_s^3} \qquad (1)$$

where

$$r_s = \left(x_{1_s}^2 + x_{2_s}^2 + x_{3_s}^2\right)^{\frac{1}{2}}$$

μ_s = the product of the universal gravitational constant and the mass of the sun

$= 1.3255 \times 10^{20}\ m^3/sec^2$

$x_{1_s}, x_{2_s}, x_{3_s}$ = the position coordinates of the spacecraft with respect to the 1_s, 2_s, 3_s coordinate system.

This mathematical model assumes that the spacecraft is on an interplanetary trajectory and that the only force acting upon it is the sun's gravitational attraction. If the spacecraft were in a planetary orbit, the model would have to be modified accordingly. It should be noted that the differential equations (1) merely give an approximate description of the spacecraft motion, but are sufficient as a model for the purposes of applying optimal estimation and control theory to the system design (see Secs. III and IV).

The equations of motion (1) can be put into state-variable form upon definition of the six-dimensional state vector of the spacecraft,

$$x_s = \begin{bmatrix} x_{1_s} \\ x_{2_s} \\ x_{3_s} \\ \dot{x}_{1_s} \\ \dot{x}_{2_s} \\ \dot{x}_{3_s} \end{bmatrix} , \tag{2}$$

which enables the differential equations (1) to be rewritten concisely as

$$\dot{x}_s = \begin{bmatrix} \dot{x}_{1_s} \\ \dot{x}_{2_s} \\ \dot{x}_{3_s} \\ -\dfrac{\mu_s x_{1_s}}{r_s^3} \\ -\dfrac{\mu_s x_{2_s}}{r_s^3} \\ -\dfrac{\mu_s x_{3_s}}{r_s^3} \end{bmatrix} = f_s(x_s) \tag{3}$$

where

$f_s(x_s)$ is a six-dimensional vector function of the state x_s.

In order to account for actual input disturbances as well as inaccuracies in the mathematical model of spacecraft motion, a random forcing term is included in Eq. (3), as follows:

$$\dot{x}_s = f_s(x_s) + w_s \tag{4}$$

where the six-dimensional vector w_s (having elements w_{i_s}) is assumed to be a white, Gaussian noise process with zero mean and covariance Q_s.

The actual trajectory of the spacecraft (i.e., the trajectory to be tracked by the telescope in the digital-computer simulation) is generated by a set of differential equations that are more complete than the model described in Eqs. (1). For an Earth-Mars mission, the forces acting upon the spacecraft due to the gravitational fields of Mars and the earth, as well as the sun, are considered. Hence, the actual equations of motion for the spacecraft are taken to be

$$\ddot{x}_{1_s} = -\frac{\mu_s x_{1_s}}{r_s^3} - \frac{\mu_m\left(x_{1_s} - x_{1_m}\right)}{r_m^3} - \frac{\mu_e\left(x_{1_s} - x_{1_e}\right)}{r_e^3}$$

$$\ddot{x}_{2_s} = -\frac{\mu_s x_{2_s}}{r_s^3} - \frac{\mu_m\left(x_{2_s} - x_{2_m}\right)}{r_m^3} - \frac{\mu_e\left(x_{2_s} - x_{2_e}\right)}{r_e^3}$$

$$\ddot{x}_{3_s} = -\frac{\mu_s x_{3_s}}{r_s^3} - \frac{\mu_m\left(x_{3_s} - x_{3_m}\right)}{r_m^3} - \frac{\mu_e\left(x_{3_s} - x_{3_e}\right)}{r_e^3} \tag{5}$$

where

$$r_m = \left[\left(x_{1_s} - x_{1_m}\right)^2 + \left(x_{2_s} - x_{2_m}\right)^2 + \left(x_{3_s} - x_{3_m}\right)^2\right]^{\frac{1}{2}}$$

$$r_e = \left[\left(x_{1_s} - x_{1_e}\right)^2 + \left(x_{2_s} - x_{2_e}\right)^2 + \left(x_{3_s} - x_{3_e}\right)^2\right]^{\frac{1}{2}}$$

μ_m = the product of the universal gravitational constant and the mass of Mars

$= 4.251 \times 10^{13}\ m^3/sec^2$

μ_e = the product of the universal gravitational constant and the mass of the earth

$= 3.986 \times 10^{14}\ m^3/sec^2$

$x_{1_m}, x_{2_m}, x_{3_m}$ = the position coordinates of Mars in the 1_s, 2_s, 3_s coordinate system

$x_{1_e}, x_{2_e}, x_{3_e}$ = the position coordinates of the earth in the 1_s, 2_s, 3_s coordinate system.

It should be noted that the x_{i_m} and x_{i_e} are well-known functions of time dictated by the orbital motions of Mars and the earth, respectively.

B. Telescope Dynamics

To describe the telescope's mechanical motion a very simplified diagram of its structure, relative to the 1_t, 2_t, 3_t coordinate system of the tracking site, is given in Fig. 3. The 1_t and 2_t axes define a

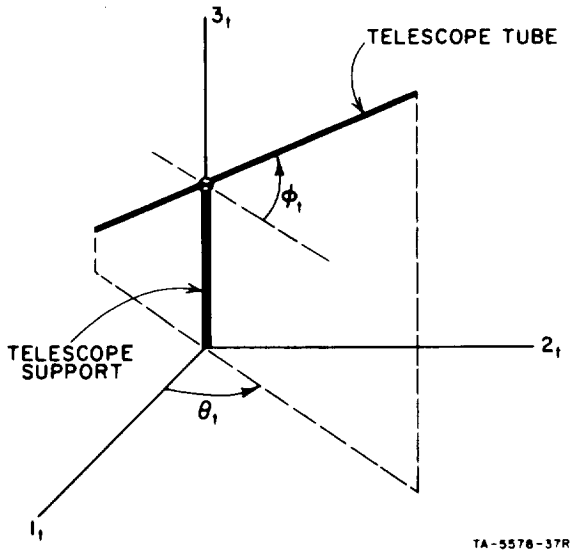

FIG. 3 MECHANICAL STRUCTURE OF TELESCOPE

plane parallel to the earth's equatorial plane, with the 1_t axis defined as lying in the plane of the great circle passing through the tracking site and the earth's poles (the 1_t axis is determined by the tracking site's longitude, which is related to the prime meridian by a constant angle); the 3_t axis is parallel to the earth's polar axis. It has been assumed that the telescope operates on an equatorial mount; hence, tracking and pointing are in terms of the telescope declination and hour angles,* φ_t and θ_t. The telescope tube is gimbal-mounted on the telescope support and rotates about the declination axis in a plane perpendicular to the plane determined by the 1_t and 2_t axes; the entire telescope structure rotates about the polar (or hour-angle) axis, which corresponds to the 3_t axis. The polar axis subtends an angle of $90^\circ - \psi_t$ with a normal to the earth at the tracking site, where ψ_t is the latitude of the site. The relative orientation of the 1_t, 2_t, 3_t and 1_s, 2_s, 3_s coordinate systems is given by the orthogonal transformation

$$T_{t/s} = \begin{bmatrix} \cos\eta \cos(\Omega t + \delta) & \sin(\Omega t + \delta) & -\sin\eta \cos(\Omega t + \delta) \\ -\cos\eta \sin(\Omega t + \delta) & \cos(\Omega t + \delta) & \sin\eta \sin(\Omega t + \delta) \\ \sin\eta & 0 & \cos\eta \end{bmatrix} \quad (6)$$

where

Ω = angular rate of rotation of earth

= 7.28×10^{-5} rad/sec

δ = arbitrary angle

η = angle between earth's polar axis and a normal to the plane of the ecliptic

= 23.45° (0.41 rad).

---

*This terminology is consistent with that used in astronomy.

A linear model to describe the mechanical behavior of the telescope and its drive motors is taken to be

$$J_{\varphi}\,\ddot{\varphi}_t + f_{\varphi}\,\dot{\varphi}_t = u_{\varphi}$$

$$J_{\theta}(t)\,\ddot{\theta}_t + f_{\theta}\,\dot{\theta}_t = u_{\theta} \qquad (7)$$

where

J_{φ} = moment of inertia of the telescope tube about the declination axis

$J_{\theta}(t) = J'_{\theta} + J_{\varphi}\cos^2\varphi_t$

= instantaneous moment of inertia of the entire telescope structure about the polar axis

J'_{θ} = moment of inertia about the polar axis of the telescope structure with the telescope tube oriented at $\varphi_t = 90^{\circ}$

f_{φ}, f_{θ} = damping factors due to the bearings that support the telescope's declination and polar axes, respectively

u_{φ}, u_{θ} = motor torques about the telescope's declination and polar axes, respectively; these quantities correspond to the control variables.

In modeling the telescope's mechanical motion, bending of the telescope structure caused by the force of gravity is not included. Since this bending (which is essentially a static effect) can be determined experimentally as a function of the angles φ_t and θ_t, it can be compensated for in the system design and need not be included in the telescope's dynamic equations. For deep-space communication (e.g., an Earth-Mars mission), the telescope will be subjected to very small angular rates and accelerations. Thus, since the telescope is an extremely rigid structure, it is reasonable to assume that any bending of the telescope structure due to these accelerations is negligible. Therefore, in the above mathematical model it has been assumed that the telescope tube and support are both perfectly rigid. Thermal distortion

of the telescope is essentially a steady-state effect (having a time constant on the order of hours) which can readily be measured and compensated for in the system. Hence it is not necessary to include this effect in the telescope's dynamic equations. Additionally, J_θ can be considered to be a function of time. It should be noted that the differential equations (7) merely give an approximate description of the telescope's mechanical motion and its drive motors, but are sufficient as a model for the purposes of applying optimal estimation and control theory to the system design (see Secs. III and IV).

It is assumed that hydrostatic bearings are used to support the telescope mount's declination and polar axes. Hydrostatic bearings are employed to obtain a high degree of smoothness in the mechanical motion of the telescope and, hence, to permit the development of a high-precision pointing and tracking system. Furthermore, these bearings reduce the friction to exceedingly small values, so that the telescope can be driven by relatively small motors for the low tracking rates that are required. With hydrostatic bearings, the breakaway torque (static friction) is negligible, and it is not necessary to include this effect in the model for the telescope's dynamics. It is also assumed that direct-drive torque motors are employed to generate the torques (controls) u_φ and u_θ, which drive the telescope about the declination and polar axes, respectively. Torque motors are designed to provide very smooth output torque (independent of shaft position), thus making possible a high degree of resolution in position at low operating speeds. Direct-drive torque motors are attached directly to the load itself (i.e., the telescope shafts); hence, there is no gear train required, with its inherent windup and backlash errors. Furthermore, the dynamics of the drive motors (having time constants less than one millisecond) are considerably faster than the telescope's dynamic behavior and are considered as having a transfer function of unity in the above model.

It is assumed that the earth-based terminal is equipped with a 120-inch (approximately 3-meter) telescope, similar in structure to the

Lick Observatory telescope on Mt. Hamilton in California. The specifications for this telescope[2] have been used to arrive at the following nominal values for the telescope parameters defined in Eq. (7):

$$J_\varphi = 1.9 \times 10^6 \ \text{kg}\cdot\text{m}^2$$

$$J'_\theta = 0.6 \times 10^6 \ \text{kg}\cdot\text{m}^2$$

$$f_\varphi = 0.8 \times 10^5 \ \frac{\text{N}\cdot\text{m}}{\text{rad/sec}}$$

$$f_\theta = 3.1 \times 10^5 \ \frac{\text{N}\cdot\text{m}}{\text{rad/sec}} \quad . \tag{8}$$

The equations of motion (7) can be put into state-variable form by defining the four-dimensional state vector of the telescope

$$x_t = \begin{bmatrix} \varphi_t \\ \dot{\varphi}_t \\ \theta_t \\ \dot{\theta}_t \end{bmatrix} \tag{9}$$

and the two-dimensional control vector

$$u = \begin{bmatrix} u_\varphi \\ u_\theta \end{bmatrix} \quad , \tag{10}$$

which enables the differential equations (7) to be rewritten concisely as

$$\begin{aligned} \dot{x}_t &= F_t(t)\, x_t + D_t(t)\, u \\ &= \begin{bmatrix} F_\varphi & 0 \\ 0 & F_\theta(t) \end{bmatrix} x_t + \begin{bmatrix} D_\varphi & 0 \\ 0 & D_\theta(t) \end{bmatrix} u \end{aligned} \tag{11}$$

where

$$F_\varphi = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{f_\varphi}{J_\varphi} \end{bmatrix} \qquad F_\theta(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{f_\theta}{J_\theta(t)} \end{bmatrix}$$

$$D_\varphi = \begin{bmatrix} 0 \\ \frac{1}{J_\varphi} \end{bmatrix} \qquad D_\theta(t) = \begin{bmatrix} 0 \\ \frac{1}{J_\theta(t)} \end{bmatrix} .$$

Equation (11) indicates the uncoupled behavior of the telescope's declination and polar channels. This model permits the computation of the optimal control law to be greatly simplified, as will be shown in Sec. IV.

Since a digital computer is to be employed for the purpose of controlling the telescope, the differential equation (11) will be converted into an equivalent difference equation. This is accomplished by assuming that u, as well as F_θ and D_θ, is constant over the interval $[k\Delta t, (k + 1)\Delta t]$:

$$x_t(k + 1) = \Phi_t(k)\, x_t(k) + \Delta_t(k)\, u(k)$$

$$= \begin{bmatrix} \Phi_\varphi & 0 \\ 0 & \Phi_\theta(k) \end{bmatrix} x_t(k) + \begin{bmatrix} \Delta_\varphi & 0 \\ 0 & \Delta_\theta(k) \end{bmatrix} u(k) \qquad (12)$$

where

$$\Phi_\varphi = \sum_{i=0}^{\infty} \frac{F_\varphi^i \cdot (\Delta t)^i}{i!}$$

$$\Phi_\theta(k) = \sum_{i=0}^{\infty} \frac{F_\theta^i(k) \cdot (\Delta t)^i}{i!}$$

$$\Delta_{\varphi} = \left[\sum_{i=0}^{\infty} \frac{F_{\varphi}^{i} \cdot (\Delta t)^{i+1}}{(i+1)!}\right] D_{\varphi}$$

$$\Delta_{\theta}(k) = \left[\sum_{i=0}^{\infty} \frac{F_{\theta}^{i}(k) \cdot (\Delta t)^{i+1}}{(i+1)!}\right] D_{\theta}(k) \qquad (13)$$

where $x_t(k)$ and $u(k)$ are abbreviated representations for $x_t(k\Delta t)$ and $u(k\Delta t)$, respectively (Δt is the sampling interval). The matrices Φ_t and Δ_t can be computed with an arbitrary degree of accuracy by taking a suitably large, but finite, number of terms in the series expansions of Eqs. (13).

In order to account for actual input disturbances as well as inaccuracies in the mathematical model of the mechanical behavior of the telescope and its drive motors, a random forcing term is included in Eq. (12) as follows:

$$x_t(k+1) = \Phi_t(k)\, x_t(k) + \Delta_t(k)\, u(k) + w_t(k) \qquad (14)$$

where the four-dimensional vector w_t (having elements w_{i_t}) is assumed to be a white, Gaussian noise process with zero mean and covariance Q_t. It is also assumed that the telescope is enclosed in a dome so that it is not subjected to wind disturbances, but such load disturbances can easily be included in this formulation.

The actual mechanical behavior of the telescope and its drive motors (i.e., the telescope's mechanical motion in the digital computer simulation) is generated by a set of differential equations that are more complete than the model described in Eqs. (7). The actual equations of motion for the telescope and its drive motors are taken to be

$$J_{\varphi}\, \ddot{\varphi}_t + f_{\varphi}\, \dot{\varphi}_t + J_{\varphi}\, \dot{\theta}_t^2 \cos \varphi_t \sin \varphi_t = u_{\varphi}$$

$$\left(J_{\theta}' + J_{\varphi} \cos^2 \varphi_t\right) \ddot{\theta}_t + f_{\theta}\, \dot{\theta}_t - 2J_{\varphi}\, \dot{\theta}_t\, \dot{\varphi}_t \cos \varphi_t \sin \varphi_t = u_{\theta} \quad . \qquad (15)$$

The nonlinear terms in the above differential equations are introduced by the motion of the telescope about its declination and polar axes. For very small tracking rates these nonlinear terms will be negligible.

C. Measurement System

The measurement system consists of the following elements.

Digital Angular Readouts: These are employed to measure φ_t and θ_t, the angular positions of the telescope's declination and polar (or hour-angle) axes, respectively. These observations are corrupted by the measurement noises v_{φ_t} and v_{θ_t}, which are modeled as white, Gaussian noise processes with zero means and variances denoted by $\sigma^2_{\varphi_t}$ and $\sigma^2_{\theta_t}$. (Actually, these measurement noises have discrete probability densities.) In addition, v_{φ_t} and v_{θ_t} are statistically independent.

Optical Detector: This measures the angular pointing errors, $\varphi_0 - \varphi_s'$ and $\theta_0 - \theta_s'$, where φ_0 and θ_0 are the declination and hour angles of the telescope's optical axis, respectively, and φ_s' and θ_s' are the declination and hour angles of the optical signal from the spacecraft. The terms defining the instantaneous angle of arrival of the signal are given by $\varphi_s' = \varphi_s + v_{\varphi_a}$ and $\theta_s' = \theta_s + v_{\theta_a}$, where φ_s and θ_s are the spacecraft's declination and hour angles, respectively (these angles, which determine the geometric angle of arrival of the signal to be tracked, are functions of x_s and will be defined below), and v_{φ_a} and v_{θ_a} represent the effect of atmospheric-induced image excursion, which is a relatively rapid refractive bending of the optical beam (it is not necessary to include the effect of the steady-state refraction, since it can be accurately measured and compensated for in the system design). The random disturbances v_{φ_a} and v_{θ_a} can be modeled as white,* Gaussian noise processes with zero means and variances given by $\sigma^2_{\varphi_a}$ and $\sigma^2_{\theta_a}$.

---

*For the purposes of estimation this is a reasonable approximation, since the power spectral densities of v_{φ_a} and v_{θ_a} have bandwidths of about 250 rad/sec, which is appreciably greater than the bandwidth of the telescope's mechanical motion (0.05 to 0.50 rad/sec).

Additionally, v_{φ_a} and v_{θ_a} are assumed to be statistically independent. It is assumed that a charge-storage tube (such as an iconoscope or an image orthicon) employing a raster scan is used as the optical detector; in a tracking mode, the pointing errors are small enough so that the operation of the detector can be considered linear. Other detection techniques, such as a tetrahedral beam-splitting prism or conical scanning, could also be employed. The observations of the angular pointing errors are corrupted by the measurement noises v_{φ_d} and v_{θ_d}, which can be modeled as white, Gaussian noise processes with zero means and variances denoted by $\sigma^2_{\varphi_d}$ and $\sigma^2_{\theta_d}$. Furthermore, v_{φ_d} and v_{θ_d} are uncorrelated.

Coherent Code Transponder on Spacecraft: This is employed to measure the spacecraft range ρ_s (this quantity, which is a function of x_s, will be defined below). This measurement can be obtained by use of the primary optical channel or an auxiliary microwave link. The observation of the spacecraft range is corrupted by the measurement noise v_ρ, which is modeled as a white, Gaussian noise process with zero mean and variance given by σ^2_ρ.

The development of the above mathematical models for the statistics of the measurement system are presented in greater detail in Ref. 2.

The spacecraft's declination and hour angles, φ_s and θ_s, are defined in the same way as the corresponding telescope angles, which are shown in Fig. 3. The expressions for φ_s and θ_s are obtained as follows: In the 1_s, 2_s, 3_s coordinate system (see Fig. 2), the three-dimensional vector defining the geometric line of sight from the tracking site to the spacecraft is given by

$$z_s = x'_s - y_s \tag{16}$$

where

$$x'_s = \begin{bmatrix} x_{1_s} \\ x_{2_s} \\ x_{3_s} \end{bmatrix} = \text{the spacecraft position}$$

and y_s, which is the position of the earth-based tracking site, is the sum of two components: the earth's orbital motion and the earth's rotation about its axis--both of which are well-known functions of time. Using the orthogonal transformation of Eq. (6), z_s can be expressed in the 1_t, 2_t, 3_t coordinate system as

$$z_t = T_{t/s}\, z_s \quad . \tag{17}$$

Then it can be shown that

$$\varphi_s = \sin^{-1}\left[\frac{z_{3_t}}{\left(z_{1_t}^2 + z_{2_t}^2 + z_{3_t}^2\right)^{\frac{1}{2}}}\right] \tag{18}$$

and

$$\theta_s = \sin^{-1}\left[\frac{z_{2_t}}{\left(z_{1_t}^2 + z_{2_t}^2\right)^{\frac{1}{2}}}\right] \quad . \tag{19}$$

The spacecraft range ρ_s, which is the distance from the tracking site to the spacecraft, is given by

$$\rho_s = |z_t| = |z_s| \quad . \tag{20}$$

By substituting from Eqs. (16) and (17), one can express φ_s, θ_s, and ρ_s in terms of the x_i, which are contained in the spacecraft state vector [Eq. (2)], and the elements of y, which are known time-varying functions describing the earth's orbital and rotational motion.

The system described in Fig. 1 consists of two tracking loops operating in series. The telescope's mechanical axis and its drive motors are located in the primary tracking loop, together with the estimator and controller. The function of the primary loop is to provide smooth tracking of φ_s and θ_s, the slowly varying mean value of the

incoming signal's angle of arrival, by the telescope's mechanical axis, which is defined by φ_t and θ_t. By providing smooth tracking of the angle-of-arrival mean value, the signal's instantaneous angle of arrival (defined by $\varphi_s + v_{\varphi_a}$ and $\theta_s + v_{\theta_a}$) will be kept within the dynamic range of the beam steerer, which is in the secondary loop. The secondary tracking loop is a direct feedback of the optical detector output to a small-angle, high-frequency beam steerer. The purpose of this secondary tracking loop is to cause the telescope's optical axis (defined by φ_0 and θ_0) to follow the rapid fluctuations in the signal's angle of arrival, thus ensuring the maximum possible signal-to-noise ratio. This is achieved by nulling the angular pointing errors.

The tracking configuration for the declination channel is shown in Fig. 4; the polar (or hour-angle) channel has the identical form. It is assumed that the beam steerer is implemented by a system of piezoelectrically-rotated mirrors, which provide orthogonal deflections

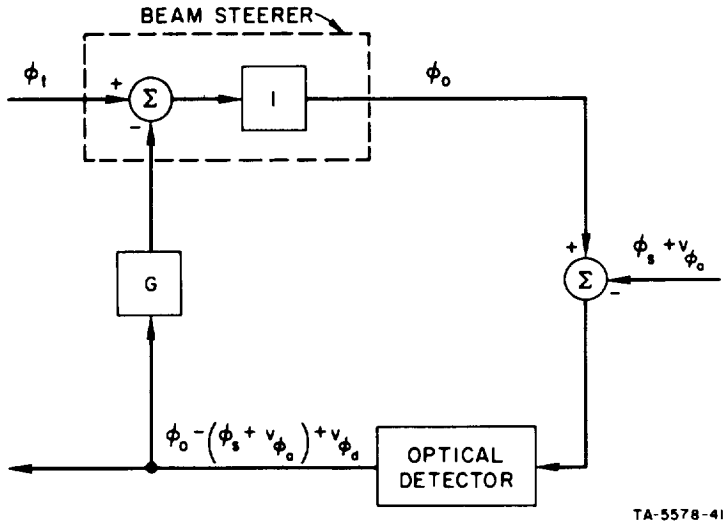

FIG. 4 TRACKING CONFIGURATION (declination channel)

of the optical beam about the declination and polar axes. This type of beam steerer can be designed to have a bandwidth of about 20 to 50 kHz, and be essentially linear over a dynamic range of ±100 seconds of arc.

Since the beam steerer has a bandwidth that is considerably greater than the bandwidth of the telescope's mechanical motion, it is reasonable to consider it to be described by a transfer function of unity. The feedback gain G (where $G > 0$) is chosen sufficiently large, consistent with stability considerations, to reduce the angular pointing error to an acceptable level.

From Fig. 4, it can be shown that the output of the optical detector for the declination channel is given by

$$\varphi_0 - \left(\varphi_s + v_{\varphi_a}\right) + v_{\varphi_d} = \frac{1}{1 + G}\left[\varphi_t - \left(\varphi_s + v_{\varphi_a}\right) + v_{\varphi_d}\right] \quad . \quad (21)$$

Similarly, the output of the optical detector for the polar channel is

$$\theta_0 - \left(\theta_s + v_{\theta_a}\right) + v_{\theta_d} = \frac{1}{1 + G}\left[\theta_t - \left(\theta_s + v_{\theta_a}\right) + v_{\theta_d}\right] \quad . \quad (22)$$

The equations for the measurement system can be expressed concisely by defining the 10-dimensional state vector x of the overall system, which consists of the spacecraft state x_s and the telescope state x_t:

$$x = \begin{bmatrix} x_s \\ \\ x_t \end{bmatrix} \quad . \quad (23)$$

Hence, the measurement equations can be defined in terms of the state vector x by the five-dimensional measurement vector

$$\beta(k) = \begin{bmatrix} \varphi_t(k) - \varphi_s(k) \\ \theta_t(k) - \theta_s(k) \\ \rho_s(k) \\ \varphi_t(k) \\ \theta_t(k) \end{bmatrix} + \begin{bmatrix} v_{\varphi_d}(k) - v_{\varphi_a}(k) \\ v_{\theta_d}(k) - v_{\theta_a}(k) \\ v_\rho(k) \\ v_{\varphi_t}(k) \\ v_{\theta_t}(k) \end{bmatrix} = h[x(k), k] + v(k) \quad . \quad (24)$$

From Eqs. (18), (19), and (20), the spacecraft's declination and hour angles and the spacecraft range are time-varying, nonlinear functions of the system state--$\varphi_s[x(k), k]$, $\theta_s[x(k), k]$, and $\rho_s[x(k), k]$. Hence, the measurement equations are time-varying and nonlinear as represented by $h[x(k), k]$. From the above discussion, the five-dimensional measurement noise vector $v(k)$ is a white, Gaussian noise process with

$$E[v(k)] = 0$$

$$E[v(k)\, v^T(k)] = R(k) = \begin{bmatrix} \sigma^2_{\varphi_d}(k) + \sigma^2_{\varphi_a}(k) & & & & \\ & \sigma^2_{\theta_d}(k) + \sigma^2_{\theta_a}(k) & & & \\ & & \sigma^2_{\rho}(k) & & \\ & & & \sigma^2_{\varphi_t}(k) & \\ & & & & \sigma^2_{\theta_t}(k) \end{bmatrix} . \tag{25}$$

The covariance matrix $R(k)$ takes this form since the optical-detector noise, the noise introduced by image excursion, the noise on the range measurement, and the digital angular-readout noise are mutually independent. For the assumed system parameters,[2] the variances in $R(k)$ are given as follows:

$$\begin{aligned} \sigma^2_{\varphi_d}(k) &= \sigma^2_{\theta_d}(k) = 6 \times 10^{-19}\ \text{rad}^2 \\ \sigma^2_{\varphi_a}(k) &= \sigma^2_{\theta_a}(k) = 2.25 \times 10^{-10}\ \text{rad}^2 \\ \sigma^2_{\rho}(k) &= 10^8\ \text{m}^2 \\ \sigma^2_{\varphi_t}(k) &= \sigma^2_{\theta_t}(k) = 2.25 \times 10^{-12}\ \text{rad}^2 \end{aligned} \quad . \tag{26}$$

D. Estimator and Controller

The function of the estimator is to generate a "best" estimate of the present system state x, based on all measurements β (which are corrupted by noise) up to the present time. This estimate is then employed in the controller to compute a "best" control u with respect to a given performance criterion. The estimation and control equations are derived in Secs. III and IV, respectively; these derivations closely follow those in Memorandum 7,[4] which describes the problem of high-precision tracking of space vehicles by large radio antennas. The approach taken assumes that the equations of the estimator and the controller can be solved separately. Since the optical-tracking problem is nonlinear, this assumption may yield a solution that is not strictly optimal.[3] However, this approach enables one to obtain a solution that is computationally feasible.

III ESTIMATION EQUATIONS

In this section the equations for the estimator are developed by employing the extended Kalman filter. This concept is an application to nonlinear systems of work done by Kalman[5] in linear estimation theory, in which the estimate obtained at each time is the maximum likelihood estimate conditioned on all measurements up to that time. Justification for this approach and a derivation of the extended (or linearized) Kalman filter is presented in Memorandum 6[6] and, hence, will not be repeated here.

From Eqs. (4) and (14), the random disturbance in the model of the overall system is given by the ten-dimensional vector

$$w(k) = \begin{bmatrix} w_s(k) \\ \\ w_t(k) \end{bmatrix} \quad ,$$

which is a white, Gaussian noise process with

$$E[w(k)] = 0$$

$$E[w(k)\ w^T(k)] = Q(k) = \begin{bmatrix} Q_s(k) & 0 \\ \\ 0 & Q_t(k) \end{bmatrix} \quad . \qquad (27)$$

A model for the measurement noise v(k) is given in Eqs. (25). The initial state x(0) is taken to be a Gaussian random variable with

$$E[x(0)] = \hat{x}(0/0)$$

$$E[\{x(0) - \hat{x}(0/0)\}\ \{x(0) - \hat{x}(0/0)\}^T] = P(0/0) \quad . \qquad (28)$$

Furthermore, it is assumed that w(k), v(k), and x(0) are uncorrelated.

The resulting estimation equations can be considered as consisting of two parts: prediction and correction (or regression). The following notation will be employed:

$$\hat{x}(i/j) \triangleq E[x(i)/\beta(j),\ldots,\beta(1),u(j-1),\ldots,u(0)] \quad ,$$

$$P(i/j) \triangleq E[\{x(i) - \hat{x}(i/j)\}\{x(i) - \hat{x}(i/j)\}^T/\beta(j),\ldots,\beta(1),u(j-1),\ldots,u(0)] \ .$$

These expectations are conditioned on the previous measurements and inputs.

A. Prediction

Given the estimate of the system state at the k^{th} instant $[\hat{x}(k/k)]$, and the assumption that $E[w(k)] = 0$, the predicted system state for the $k+1^{th}$ instant $[\hat{x}(k+1/k)]$ is obtained from Eqs. (4) and (14):*

$$\hat{x}(k+1/k) = \begin{bmatrix} \hat{x}_s(k+1/k) \\ \\ \hat{x}_t(k+1/k) \end{bmatrix} = \begin{bmatrix} \hat{x}_s(k/k) + f_s[\hat{x}_s(k/k)]\Delta t \\ \\ \Phi_t(k)\ \hat{x}_t(k/k) + \Delta_t(k)\ u(k) \end{bmatrix} \quad . \quad (29)$$

The covariance of the error in this prediction is given by

$$P(k+1/k) = \Phi(k)\ P(k/k)\ \Phi^T(k) + Q(k) \qquad (30)$$

where

$$\Phi(k) = \begin{bmatrix} \Phi_s(k) & 0 \\ \\ 0 & \Phi_t(k) \end{bmatrix} \qquad (31)$$

---

*The nonlinear differential equation for x_s may be integrated by a more accurate method if necessary.

with $\Phi_t(k)$ defined in Eq. (12) and $\Phi_s(k)$ obtained from the nonlinear differential equation (3) by linearization about the estimate $\hat{x}(k/k)$ [or $\hat{x}_s(k/k)$]; i.e.,

$$\Phi_s(k) = I + \frac{\partial f_s}{\partial x_s}\,[\hat{x}_s(k/k)]\,\Delta t \qquad (32)$$

in which

$$\frac{\partial f_s}{\partial x_s} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \dfrac{\mu_s\left(2x_{1_s}^2 - x_{2_s}^2 - x_{3_s}^2\right)}{r_s^5} & \dfrac{3\mu_s x_{1_s} x_{2_s}}{r_s^5} & \dfrac{3\mu_s x_{1_s} x_{3_s}}{r_s^5} & 0 & 0 & 0 \\ \dfrac{3\mu_s x_{2_s} x_{1_s}}{r_s^5} & \dfrac{\mu_s\left(2x_{2_s}^2 - x_{1_s}^2 - x_{3_s}^2\right)}{r_s^5} & \dfrac{3\mu_s x_{2_s} x_{3_s}}{r_s^5} & 0 & 0 & 0 \\ \dfrac{3\mu_s x_{3_s} x_{1_s}}{r_s^5} & \dfrac{3\mu_s x_{3_s} x_{2_s}}{r_s^5} & \dfrac{\mu_s\left(2x_{3_s}^2 - x_{1_s}^2 - x_{2_s}^2\right)}{r_s^5} & 0 & 0 & 0 \end{bmatrix} .$$

B. Correction

The prediction $\hat{x}(k + 1/k)$ is then corrected by using the actual measurement at the $k + 1^{th}$ instant $[\beta(k + 1)]$ and the predicted measurement for the $k + 1^{th}$ instant $[\hat{\beta}(k + 1/k)]$, which is obtained from Eq. (24) by substituting $\hat{x}(k + 1/k)$ and using the assumption that $E[v(k + 1)] = 0$:

$$\hat{\beta}(k + 1/k) = h[\hat{x}(k + 1/k),\ k + 1] \quad . \qquad (33)$$

Hence, the estimate of the system state at the $k + 1^{th}$ instant is given by

$$\hat{x}(k+1/k+1) = \begin{bmatrix} \hat{x}_s(k+1/k+1) \\ \\ \hat{x}_t(k+1/k+1) \end{bmatrix} = \hat{x}(k+1/k) + W(k+1)\ [\beta(k+1) - \hat{\beta}(k+1/k)] \quad (34)$$

where the 10 × 5 weighting matrix is given by

$$W(k+1) = P(k+1/k)\ H^T(k+1)\ [R(k+1) + H(k+1)\ P(k+1/k)\ H^T(k+1)]^{-1} \quad , \quad (35)$$

and $H(k+1)$ is obtained from Eq. (24) by linearization about the prediction $\hat{x}(k+1/k)$ [or $\hat{x}_s(k+1/k)$]; i.e.,

$$H(k+1) = H[\hat{x}(k+1/k),\ k+1] = \frac{\partial h}{\partial x}\ [\hat{x}(k+1/k),\ k+1] \quad (36)$$

in which

$$\frac{\partial h}{\partial x} = \begin{bmatrix} a_1 & a_2 & a_3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ c_1 & c_2 & c_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

with

$$a_i = -\frac{\partial \varphi_s}{\partial x_{i_s}} \quad , \quad b_i = -\frac{\partial \theta_s}{\partial x_{i_s}} \quad , \quad \text{and} \quad c_i = \frac{\partial \rho_s}{\partial x_{i_s}} \quad .$$

Applying the chain rule for differentiation to Eqs. (16) to (19), it can be shown that the a_i and b_i are given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = - T_{t/s}^{T} \begin{bmatrix} \partial\varphi_s/\partial z_{1_t} \\ \partial\varphi_s/\partial z_{2_t} \\ \partial\varphi_s/\partial z_{3_t} \end{bmatrix} \tag{37}$$

where

$$\frac{\partial\varphi_s}{\partial z_{1_t}} = \frac{- z_{1_t} z_{2_t}}{\left(z_{1_t}^2 + z_{2_t}^2\right)^{\frac{1}{2}} |z_t|^2}$$

$$\frac{\partial\varphi_s}{\partial z_{2_t}} = \frac{- z_{2_t} z_{3_t}}{\left(z_{1_t}^2 + z_{2_t}^2\right)^{\frac{1}{2}} |z_t|^2}$$

$$\frac{\partial\varphi_s}{\partial z_{3_t}} = \frac{\left(z_{1_t}^2 + z_{2_t}^2\right)^{\frac{1}{2}}}{|z_t|^2} \quad ,$$

and

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = - T_{t/s}^{T} \begin{bmatrix} \partial\theta_s/\partial z_{1_t} \\ \partial\theta_s/\partial z_{2_t} \\ 0 \end{bmatrix} \tag{38}$$

where

$$\frac{\partial \theta_s}{\partial z_{1_t}} = \frac{-z_{2_t}}{z_{1_t}^2 + z_{2_t}^2}$$

$$\frac{\partial \theta_s}{\partial z_{2_t}} = \frac{z_{1_t}}{z_{1_t}^2 + z_{2_t}^2} \quad .$$

The $\partial \varphi_s / \partial z_{j_t}$ and $\partial \theta_s / \partial z_{j_t}$ can be rewritten in terms of the x_{i_s} and the elements of y by substitution from Eqs. (16) and (17).

From Eq. (20), it is a straightforward matter to show that the c_i are given by

$$c_i = \frac{x_{i_s} - y_{i_s}}{\left[\sum_{j=1}^{3} \left(x_{j_s} - y_{j_s}\right)^2\right]^{\frac{1}{2}}} \quad . \tag{39}$$

The covariance of the error in the estimate $\hat{x}(k + 1/k + 1)$ is

$$P(k+1/k+1) = [I - W(k+1)H(k+1)]P(k+1/k)$$

$$= P(k+1/k) - P(k+1/k)H^T(k+1)[R(k+1) + H(k+1)P(k+1/k)H^T(k+1)]^{-1}$$

$$\cdot H(k+1)P(k+1/k) \quad . \tag{40}$$

The extended Kalman filter [Eqs. (29), (30), (34), (35), and (40)] can be readily implemented on a digital computer. The a priori state estimate $\hat{x}(0/0)$ and its covariance P(0/0) are used to initialize these recursive equations. It should be noted that in the extended Kalman filter, the nonlinear equations (3) and (22) are used in Eqs. (29) and (34) to obtain the predicted state and the predicted measurement. The

linearization of Eqs. (3) and (22), in order to obtain Φ_s and H, is only employed to calculate the covariance matrices P and the weighting matrix W.

Since the overall system is nonlinear, the above solution to the estimation problem is suboptimal. Intuitively, this approach seems to be quite reasonable, but its validity has not been rigorously established. The extent to which this solution to the estimation problem differs from the optimum is mainly dependent upon the accuracy of the linearization of Eq. (3), the differential equation for x_s, and of Eq. (22), the measurement equation.

IV CONTROL EQUATIONS

In this section the control problem is solved by application of linear optimal control theory. Consider the performance criterion

$$J = E\left[\sum_{k=0}^{N} \left\{[\varphi_t(k) - \varphi_s(k)]^2 + [\theta_t(k) - \theta_s(k)]^2 + b_c\left[u_\varphi^2(k) + u_\theta^2(k)\right]\right\}\right] \quad . \quad (41)$$

The cost associated with control (where $b_c \geq 0$) is essential in order to guarantee that u_φ and u_θ (the motor torques) do not become too large, which, in turn, would cause the telescope rates to exceed their permissible ranges of values. However, the actual tracking performance is given by the first two terms of Eq. (41).

To use the results of linear optimal control theory, it is necessary for the performance criterion J to be quadratic in the system state x. However, this condition is not satisfied, since φ_s and θ_s are nonlinear functions of x (or x_s), as shown by Eqs. (18) and (19). The criterion J can be put into the appropriate form by linearization of $\varphi_s(k)$ and $\theta_s(k)$ about the estimate $\hat{x}(k/k)$ [or $\hat{x}_s(k/k)$]:

$$\varphi_s(k) = \hat{\varphi}_s(k/k) + \underline{a}^T(k)\, \tilde{x}_s(k)$$

$$\theta_s(k) = \hat{\theta}_s(k/k) + \underline{b}^T(k)\, \tilde{x}_s(k) \qquad (42)$$

where

$$x_s(k) = \hat{x}_s(k/k) + \tilde{x}_s(k)$$

$$\hat{\varphi}_s(k/k) = \varphi_s[\hat{x}_s(k/k),\ k]$$

$$\hat{\theta}_s(k/k) = \theta_s[\hat{x}_s(k/k),\ k] \qquad (43)$$

and

$$\underline{a}(k) = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\hat{x}_s(k/k),\ k} \qquad \underline{b}(k) = \begin{bmatrix} -b_1 \\ -b_2 \\ -b_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\hat{x}_s(k),\ k} ,$$

in which the a_i and b_i are given by Eqs. (37) and (38). It should be noted that because of the nonlinearity of Eqs. (18) and (19), $\hat{\varphi}_s$ and $\hat{\theta}_s$ are not optimal estimates in the usual sense.

The state of the system can now be defined by the 12-dimensional vector

$$\underline{x} = \begin{bmatrix} \hat{\varphi}_s \\ \hat{\theta}_s \\ \tilde{x}_s \\ x_t \end{bmatrix} , \tag{44}$$

which contains x (with x_s linearized) and is augmented by $\hat{\varphi}_s$ and $\hat{\theta}_s$. It can be shown that the equations of motion for $\underline{x}$ are given by

$$\underline{x}(k+1) = \underline{\Phi}(k)\ \underline{x}(k) + \underline{\Delta}(k)\ u(k) + \underline{w}(k) \tag{45}$$

where

$$\underline{\Phi}(k) = \begin{bmatrix} \Phi_l(k) & 0 \\ 0 & \Phi_t(k) \end{bmatrix}$$

$$\Phi_1(k) = \begin{bmatrix} I & 0 \\ 0 & \Phi_s(k) \end{bmatrix}$$

$$\underline{\Delta}(k) = \begin{bmatrix} 0 \\ \Delta_t(k) \end{bmatrix}$$

$$\underline{w}(k) = \begin{bmatrix} 0 \\ w(k) \end{bmatrix} \quad .$$

The matrices $\Phi_t(k)$ and $\Delta_t(k)$ are defined in Eq. (12); $\Phi_s(k)$ is given in Eq. (32); the random disturbance w(k) is defined in Eq. (27).

Substituting Eqs. (42) into Eq. (41) and rewriting J according to the standard formulation gives

$$J = E\left[\sum_{k=0}^{N} \{\underline{x}^T(k)\ A(k)\ \underline{x}(k) + u^T(k)\ B\ u(k)\}\right] \tag{46}$$

where

$$B = \begin{bmatrix} b_c & 0 \\ 0 & b_c \end{bmatrix}$$

$$A(k) = \begin{bmatrix} A_1(k) & A_3(k) \\ A_3^T(k) & A_2 \end{bmatrix} \quad ,$$

in which $A_1(k)$ is 8×8, A_2 is 4×4, and $A_3(k)$ is 8×4; A(k) is symmetric and positive semidefinite. It is a straightforward matter to determine A(k):

$$A_1(k) = \left[\begin{array}{cc|c} 1 & 0 & -\underline{a}^T(k) \\ 0 & 1 & -\underline{b}^T(k) \\ \hline -\underline{a}(k) & -\underline{b}(k) & \underline{a}(k)\underline{a}^T(k) + \underline{b}(k)\underline{b}^T(k) \end{array}\right]$$

$$A_2 = \begin{bmatrix} 1 & & & \\ & 0 & \bigcirc & \\ & \bigcirc & 1 & \\ & & & 0 \end{bmatrix}$$

$$A_3(k) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \vdots & -1 & \vdots \\ -\underline{a}(k) & 0 & -\underline{b}(k) & 0 \end{bmatrix} .$$

The design objective is to find the sequence of controls

$$u(0),\ u(1),\ \ldots,\ u(N)$$

that minimizes J. The control equations will be derived by applying some results obtained by Larson, which are described in Quarterly Report 3;[7] this work is an extension of results in linear optimal control theory.

The optimal control is given by

$$u(k) = \begin{bmatrix} u_\varphi(k) \\ \\ u_\theta(k) \end{bmatrix} = -\,K(k)\ \hat{\underline{x}}(k/k) \qquad (47)$$

where the 2 × 12 gain matrix is given by

$$K(k) = \left[B + \underline{\Delta}^T(k)\ P_c(k+1)\ \underline{\Delta}^T(k)\right]^{-1}\ \underline{\Delta}^T(k)\ P_c(k+1)\ \underline{\Phi}(k) \qquad (48)$$

and the 12 × 12 matrix P_c satisfies the discrete Riccati equation

$$P_c(k) = A(k) + \underline{\Phi}^T(k)\, P_c(k+1)\, \underline{\Phi}(k) - \underline{\Phi}^T(k)\, P_c(k+1)\, \underline{\Delta}(k) \left[B + \underline{\Delta}^T(k)\, P_c(k+1)\, \underline{\Delta}(k)\right]^{-1} \cdot \underline{\Delta}^T(k)\, P_c(k+1)\, \underline{\Phi}(k) \qquad 0 \le k < N \quad , \tag{49}$$

$$P_c(N) = A(N) \quad .$$

For convenience, $P_c(k)$ will be rewritten in a form entirely analogous to A(k):

$$P_c(k) = \begin{bmatrix} P_1(k) & P_3(k) \\ P_3^T(k) & P_2(k) \end{bmatrix}$$

where $P_c(k)$ is symmetric and positive semidefinite. Upon performance of the indicated matrix multiplications, the optimal control in Eq. (47) becomes

$$u(k) = -\left[B + \Delta_t^T(k)\, P_2(k+1)\, \Delta_t(k)\right]^{-1} \Delta_t^T(k)\, P_3^T(k+1)\, \Phi_1(k)\, \hat{\underline{\Psi}}(k/k) - \left[B + \Delta_t^T(k)\, P_2(k+1)\, \Delta_t(k)\right]^{-1} \Delta_t^T(k)\, P_2(k+1)\, \Phi_t(k)\, \hat{x}_t(k/k) \tag{50}$$

where

$$\hat{\underline{\Psi}}(k/k) = \begin{bmatrix} \hat{\varphi}_s(k/k) \\ \hat{\theta}_s(k/k) \\ \hat{\tilde{x}}_s(k/k) \end{bmatrix} \quad . \tag{51}$$

The Riccati equation (49) can be partitioned into separate equations for P_1, P_2, and P_3:

$$P_1(k) = A_1(k) + \Phi_1^T(k)P_1(k+1)\Phi_1(k)$$
$$- \Phi_1^T(k)P_3^T(k+1)\Delta_t(k)\left[B + \Delta_t^T(k)P_2(k+1)\Delta_t(k)\right]^{-1}\Delta_t^T(k)P_3(k+1)\Phi_1(k)$$
$$0 \le k < N \quad , \qquad (52)$$

$$P_1(N) = A_1(N) \quad ;$$

$$P_2(k) = A_2 + \Phi_t^T(k)P_2(k+1)\Phi_t(k)$$
$$- \Phi_t^T(k)P_2(k+1)\Delta_t(k)\left[B + \Delta_t^T(k)P_2(k+1)\Delta_t(k)\right]^{-1}\Delta_t^T(k)P_2(k+1)\Phi_t(k)$$
$$0 \le k < N \quad , \qquad (53)$$

$$P_2(N) = A_2 \quad ;$$

and

$$P_3(k) = A_3(k) + \Phi_1^T(k)P_3(k+1)\Phi_t(k)$$
$$- \Phi_1^T(k)P_3(k+1)\Delta_t(k)\left[B + \Delta_t^T(k)P_2(k+1)\Delta_t(k)\right]^{-1}\Delta_t^T(k)P_2(k+1)\Phi_t(k)$$
$$0 \le k < N \quad , \qquad (54)$$

$$P_3(N) = A_3(N) \quad .$$

Equation (53) can be solved for the 4 × 4 matrix P_2 independently of Eqs. (52) and (54). It is interesting to note that Eq. (53) is the Riccati equation for the system of Eq. (14), with the performance criterion

$$E\left[\sum_{k=0}^{N}\left\{x_t^T(k)\ A_2 x_t(k) + u^T(k)\ Bu(k)\right\}\right] \quad .$$

Once P_2 has been found, it is substituted into Eq. (54), which is a linear equation in the 8 × 4 matrix P_3. Since the 8 × 8 matrix P_1 does not enter into the control equation (50) or the calculation of P_2 and P_3, it is not necessary to solve Eq. (52).

From the form of $\Phi_t(k)$ and $\Delta_t(k)$, which are defined in Eq. (12), and A_2, which is defined in Eq. (46), it can be shown that the solution to Eq. (53) is of the form

$$P_2(k) = \begin{bmatrix} P_\varphi(k) & 0 \\ 0 & P_\theta(k) \end{bmatrix} \quad . \tag{55}$$

This simplification permits the Riccati equation (53) to be partitioned into separate equations for the 2 × 2 matrices P_φ and P_θ:

$$P_\varphi(k) = A_\varphi + \Phi_\varphi^T P_\varphi(k+1)\Phi_\varphi - \Phi_\varphi^T P_\varphi(k+1)\Delta_\varphi\left[b_c + \Delta_\varphi^T P_\varphi(k+1)\Delta_\varphi\right]^{-1}\Delta_\varphi^T P_\varphi(k+1)\Phi_\varphi$$

$$0 \le k < N \quad , \tag{56}$$

$$P_\varphi(N) = A_\varphi \quad ;$$

and

$$P_\theta(k) = A_\theta + \Phi_\theta^T(k)P_\theta(k+1)\Phi_\theta(k) - \Phi_\theta^T(k)P_\theta(k+1)\Delta_\theta(k)\left[b_c + \Delta_\theta^T(k)P_\theta(k+1)\Delta_\theta(k)\right]^{-1}\Delta_\theta^T(k)P_\theta(k+1)\Phi_\theta(k)$$

$$0 \le k < N \quad , \tag{57}$$

$$P_\theta(N) = A_\theta \quad ;$$

where

$$A_{\varphi} = A_{\theta} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad .$$

From the form of $\Phi_1(k)$, which is defined in Eq. (45), $A_3(k)$, which is defined in Eq. (46), $P_2(k)$, which is given in Eq. (55), $\Phi_t(k)$, and $\Delta_t(k)$, it can be shown that the solution to Eq. (54) is of the form

$$P_3(k) = \left[\begin{array}{c|c} P_{3\varphi}(k) & P_{3\theta}(k) \\ \hline \multicolumn{2}{c}{P_4(k)} \end{array}\right] \tag{58}$$

where

$$P_{3\varphi}(k) = \begin{bmatrix} p^{\varphi}_{11}(k) & p^{\varphi}_{12}(k) \\ 0 & 0 \end{bmatrix}$$

$$P_{3\theta}(k) = \begin{bmatrix} 0 & 0 \\ p^{\theta}_{21}(k) & p^{\theta}_{22}(k) \end{bmatrix} \quad ,$$

and $P_4(k)$ is a 6 X 4 matrix. This further simplification permits Eq. (54) to be partitioned into separate equations for $P_{3\varphi}$ and $P_{3\theta}$:

$$P_{3\varphi}(k) = A_{3\varphi} + P_{3\varphi}(k+1)\Phi_{\varphi} - P_{3\varphi}(k+1)\Delta_{\varphi}\left[b_c + \Delta_{\varphi}^{T}P_{\varphi}(k+1)\Delta_{\varphi}\right]^{-1}\Delta_{\varphi}^{T}P_{\varphi}(k+1)\Phi_{\varphi}$$

$$0 \leq k < N \quad , \tag{59}$$

$$P_{3\varphi}(N) = A_{3\varphi} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad ;$$

and

$$P_{3\theta}(k) = A_{3\theta} + P_{3\theta}(k+1)\Phi_{\theta}(k)$$
$$- P_{3\theta}(k+1)\Delta_{\theta}(k)\left[b_c + \Delta_{\theta}^{T}(k)P_{\theta}(k+1)\Delta_{\theta}(k)\right]^{-1}\Delta_{\theta}^{T}(k)P_{\theta}(k+1)\Phi_{\theta}(k)$$
$$0 \leq k < N \quad , \qquad (60)$$

$$P_{3\theta}(N) = A_{3\theta} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \quad .$$

Since the matrix P_4 does not enter into the control equation (as will be shown below) or the calculation of $P_{3\varphi}$ and $P_{3\theta}$, it is not necessary to solve for it.

From Eq. (43) it can be seen that

$$\hat{\hat{x}}_s(k/k) = 0 \quad ;$$

hence, Eq. (51) yields

$$\underline{\hat{\Psi}}(k/k) = \begin{bmatrix} \hat{\varphi}_s(k/k) \\ \hat{\theta}_s(k/k) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad . \qquad (61)$$

Now, it can be shown that the following quantity from the first term of Eq. (50) is given by

$$P_3^T(k+1)\ \Phi_1(k)\ \underline{\hat{\Psi}}(k/k) = \begin{bmatrix} P_{3\varphi}^T(k+1) \begin{bmatrix} \hat{\varphi}_s(k/k) \\ 0 \end{bmatrix} \\ P_{3\theta}^T(k+1) \begin{bmatrix} 0 \\ \hat{\theta}_s(k/k) \end{bmatrix} \end{bmatrix} \quad . \qquad (62)$$

Finally, the two components of the control u(k) can be obtained separately as

$$u_\varphi(k) = -\left[b_c + \Delta_\varphi^T P_\varphi(k+1)\,\Delta_\varphi\right]^{-1} \Delta_\varphi^T P_{3\varphi}^T(k+1) \begin{bmatrix} \hat{\varphi}_s(k/k) \\ 0 \end{bmatrix}$$

$$- \left[b_c + \Delta_\varphi^T P_\varphi(k+1)\,\Delta_\varphi\right]^{-1} \Delta_\varphi^T P_\varphi(k+1)\,\Phi_\varphi\,\hat{x}_\varphi(k/k) \qquad (63)$$

and

$$u_\theta(k) = -\left[b_c + \Delta_\theta^T(k) P_\theta(k+1)\,\Delta_\theta(k)\right]^{-1} \Delta_\theta^T(k) P_{3\theta}^T(k+1) \begin{bmatrix} 0 \\ \hat{\theta}_s(k/k) \end{bmatrix}$$

$$- \left[b_c + \Delta_\theta^T(k) P_\theta(k+1)\,\Delta_\theta(k)\right]^{-1} \Delta_\theta^T(k) P_\theta(k+1)\,\Phi_\theta(k)\,\hat{x}_\theta(k/k)\ , \qquad (64)$$

where

$$\hat{x}_\varphi(k/k) = \begin{bmatrix} \hat{\varphi}_t(k/k) \\ \dot{\hat{\varphi}}_t(k/k) \end{bmatrix} \qquad \hat{x}_\theta(k/k) = \begin{bmatrix} \hat{\theta}(k/k) \\ \dot{\hat{\theta}}(k/k) \end{bmatrix} .$$

Thus, the computational requirements have been reduced markedly. Instead of solving the 12 × 12 Riccati equation (49) for P_c, we may perform the following steps: first solve the 2 × 2 Riccati equations (56) and (57) for P_φ and P_θ; then, using P_φ and P_θ, calculate $P_{3\varphi}$ and $P_{3\theta}$ from the 2 × 2 linear equations (59) and (60). The optimal control u is then obtained by substituting these matrices, together with the state estimate $\hat{x}(k/k)$ [$\hat{\varphi}_s(k/k)$ and $\hat{\theta}_s(k/k)$ are computed from Eqs. (43)], into Eqs. (63) and (64).

A. Steady-State Approximation

Suppose that the discrete equation (12), which describes the telescope dynamics, is stationary (i.e., the matrices Φ_t and Δ_t are constant);

this is equivalent to assuming that F_θ and D_θ of Eq. (11) are constant. This assumption is reasonable over a substantial time interval, since the rate of change of F_θ and D_θ is slow with respect to the system time constants. Additionally, it will be assumed that the summation in the performance criterion J of Eq. (41) is over an infinite time interval (i.e., $N = \infty$). This assumption is quite reasonable, since the interval of time during which the spacecraft is being tracked will be appreciably larger than the system time constants. With these two assumptions, computation of the optimal control u(k) is greatly simplified, as will be shown below. Formulation of the control problem in this manner will be referred to as the "steady-state approximation."

The Riccati equations, (56) and (57), become

$$P_\varphi = A_\varphi + \Phi_\varphi^T P_\varphi \Phi_\varphi - \Phi_\varphi^T P_\varphi \Delta_\varphi \left[b_c + \Delta_\varphi^T P_\varphi \Delta_\varphi \right]^{-1} \Delta_\varphi^T P_\varphi \Phi_\varphi \quad (65)$$

and

$$P_\theta = A_\theta + \Phi_\theta^T P_\theta \Phi_\theta - \Phi_\theta^T P_\theta \Delta_\theta \left[b_c + \Delta_\theta^T P_\theta \Delta_\theta \right]^{-1} \Delta_\theta^T P_\theta \Phi_\theta \quad . \quad (66)$$

The above are nonlinear algebraic equations in the steady-state matrices P_φ and P_θ, respectively. A possible way of solving these equations is to assume an initial solution, substitute it into the right-hand side, and evaluate a new solution; this iterative procedure is continued until successive solutions converge.

Instead of solving Eqs. (59) and (60) for the steady-state matrices $P_{3\varphi}$ and $P_{3\theta}$, respectively, consider the following quantities from the first terms of Eqs. (63) and (64):

$$P_{3\varphi}^T \begin{bmatrix} \hat{\varphi}_s(k/k) \\ \\ 0 \end{bmatrix} = \eta_\varphi(k) \quad (67)$$

and

$$P_{3\theta}^{T} \begin{bmatrix} 0 \\ \hat{\theta}_s(k/k) \end{bmatrix} = \eta_\theta(k) \quad . \tag{68}$$

Transposing Eqs. (59) and (60), multiplying by

$$\begin{bmatrix} \hat{\varphi}_s(k/k) \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ \hat{\theta}_s(k/k) \end{bmatrix} \quad ,$$

respectively, and using the definitions in Eqs. (67) and (68), yields

$$\eta_\varphi(k) = \begin{bmatrix} -\hat{\varphi}_s(k/k) \\ 0 \end{bmatrix} + \Phi_\varphi^T \eta_\varphi(k)$$

$$- \Phi_\varphi^T P_\varphi \Delta_\varphi \left[b_c + \Delta_\varphi^T P_\varphi \Delta_\varphi \right]^{-1} \Delta_\varphi^T \eta_\varphi(k) \tag{69}$$

and

$$\eta_\theta(k) = \begin{bmatrix} -\hat{\theta}_s(k/k) \\ 0 \end{bmatrix} + \Phi_\theta^T \eta_\theta(k)$$

$$- \Phi_\theta^T P_\theta \Delta_\theta \left[b_c + \Delta_\theta^T P_\theta \Delta_\theta \right]^{-1} \Delta_\theta^T \eta_\theta(k) \quad . \tag{70}$$

Solving for $\eta_\varphi(k)$ and $\eta_\theta(k)$ yields

$$\eta_\varphi(k) = [I - \Psi_\varphi]^{-1} \begin{bmatrix} -\hat{\varphi}_s(k/k) \\ 0 \end{bmatrix} \tag{71}$$

where

$$\Psi_{\varphi} = \Phi_{\varphi}^{T} - \Phi_{\varphi}^{T} P_{\varphi} \Delta_{\varphi} \left[b_c + \Delta_{\varphi}^{T} P_{\varphi} \Delta_{\varphi} \right]^{-1} \Delta_{\varphi}^{T} \quad ,$$

and

$$\eta_{\theta}(k) = [I - \Psi_{\theta}]^{-1} \begin{bmatrix} -\hat{\theta}_s(k/k) \\ 0 \end{bmatrix} \tag{72}$$

where

$$\Psi_{\theta} = \Phi_{\theta}^{T} - \Phi_{\theta}^{T} P_{\theta} \Delta_{\theta} \left[b_c + \Delta_{\theta}^{T} P_{\theta} \Delta_{\theta} \right]^{-1} \Delta_{\theta}^{T} \quad .$$

The matrices Ψ_{φ} and Ψ_{θ} correspond to the closed-loop transition matrices of the declination and polar (or hour-angle) channels of the telescope, respectively. For a control law (as given by P_{φ} and P_{θ}) that is asymptotically stable, the eigenvalues of Ψ_{φ} and Ψ_{θ} have magnitude less than unity. With this condition satisfied it can be demonstrated that the inverses of $[I - \Psi_{\varphi}]$ and $[I - \Psi_{\theta}]$ exist.

Therefore, the controls are given by

$$\begin{aligned} u_{\varphi}(k) = &- \left[b_c + \Delta_{\varphi}^{T} P_{\varphi} \Delta_{\varphi} \right]^{-1} \Delta_{\varphi}^{T} \eta_{\varphi}(k) \\ &- \left[b_c + \Delta_{\varphi}^{T} P_{\varphi} \Delta_{\varphi} \right]^{-1} \Delta_{\varphi}^{T} P_{\varphi} \Phi_{\varphi} \hat{x}_{\varphi}(k/k) \end{aligned} \tag{73}$$

and

$$\begin{aligned} u_{\theta}(k) = &- \left[b_c + \Delta_{\theta}^{T} P_{\theta} \Delta_{\theta} \right]^{-1} \Delta_{\theta}^{T} \eta_{\theta}(k) \\ &- \left[b_c + \Delta_{\theta}^{T} P_{\theta} \Delta_{\theta} \right]^{-1} \Delta_{\theta}^{T} P_{\theta} \Phi_{\theta} \hat{x}_{\theta}(k/k) \quad . \end{aligned} \tag{74}$$

Since $\hat{\varphi}_s(k/k)$ and $\hat{\theta}_s(k/k)$, the estimates of the spacecraft's declination and hour angle, change slowly with respect to the system time constants,

use of the steady-state approximation to compute the control is quite reasonable.

As a further refinement to this approximation, the time-varying nature of F_θ and D_θ can be taken into account as follows: F_θ and D_θ are periodically updated and Φ_θ and Δ_θ of Eq. (12) are recalculated. With these new matrices, P_θ [the solution to Eq. (66)] and Ψ_θ [see Eq. (72)] are recomputed. Finally, u_θ is obtained from Eq. (74) by substituting these updated matrices. Thus, a nonstationary problem is solved as a series of different stationary problems. It is not necessary to repeat this procedure at every discrete instant $k\Delta t$, since the rate of change of F_θ and D_θ is slow with respect to the system time constants.

The solutions to the control problem can be readily implemented on a digital computer. Although these solutions are suboptimal, the approach used seems quite reasonable.

V SUMMARY AND FUTURE WORK

The digital-computer program for simulating the operation of the derived estimator-controller configuration has been written. The program has been organized so that it will be sufficiently general and flexible enough for the proposed applications.

This program is a valuable study tool for the investigation of several important topics. Primarily, it will provide a way of evaluating and comparing the performance of existing optical tracking techniques (such as conventional autotrackers). An important consideration relates to the linearizations employed in Secs. III and IV in order to obtain solutions to the estimation and control problems. Since the optical-tracking problem is nonlinear, the solutions obtained in this manner are suboptimal. Although this approach is quite reasonable, its validity has not been rigorously established. The increment time Δt has a direct effect on the accuracy of the linearizations employed. To reduce the computation time, which is an important consideration for real-time applications, it is essential to choose Δt as large as possible; however, making Δt larger will degrade the system performance to some extent since the linearizations will be increasingly violated. The tradeoff between system performance and computation as Δt varies should be ascertained. A related question concerns the mathematical models used to describe the spacecraft and the telescope dynamics. For real-time applications it is necessary to use simplified models so that the computational requirements can be significantly reduced. The extent to which such models will cause a deterioration in system performance should also be studied.

REFERENCES

1. E. C. Fraser and R. M. Dressler, "Deep-Space Optical Communication Systems," Memorandum 8, Contract NAS 12-59, SRI Project 5578, Stanford Research Institute, Menlo Park, California (30 December 1966).

2. R. M. Dressler and E. C. Fraser, "Models for the Earth-Based Terminal of an Optical Communication System," Memorandum 9, Contract NAS 12-59, SRI Project 5578, Stanford Research Institute, Menlo Park, California (March 1967).

3. L. Meier, "Combined Optimum Control and Estimation Theory," Contractor Report, Contract NAS 2-2457, SRI Project 5237, Stanford Research Institute, Menlo Park, California (October 1965).

4. R. M. Dressler, "Application of Optimum Estimation and Control Theory to Satellite Tracking Problems," Memorandum 7, Contract NAS 12-59, SRI Project 5578, Stanford Research Institute, Menlo Park, California (6 April 1966).

5. R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," Trans. ASME, J. Basic Engineering, pp. 35-45 (March 1960).

6. L. Meier, "Adaptive Control and the Combined Optimization Problem," Memorandum 6, Contract NAS 12-59, SRI Project 5578, Stanford Research Institute, Menlo Park, California (10 February 1966).

7. R. E. Larson, R. M. Dressler, and J. Peschon, "Research on Adaptive Control System Design," Quarterly Report 3, Contract NAS 12-59, SRI Project 5578, Stanford Research Institute, Menlo Park, California (April 1966).

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)	2a. REPORT SECURITY CLASSIFICATION
Stanford Research Institute 333 Ravenswood Menlo Park, California 94025	Unclassified
	2b. GROUP N/A

3. REPORT TITLE

APPLICATION OF OPTIMUM ESTIMATION AND CONTROL THEORY TO OPTICAL TRACKING PROBLEMS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Quarterly Report 5 Covering the Period 15 January to 15 April 1967

5. AUTHOR(S) (First name, middle initial, last name)

Robert M. Dressler

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May 1967	48	7

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
NAS12-59 b. PROJECT NO.	SRI Project 5578
c. d.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	National Aeronautics and Space Administration Electronics Research Center Cambridge, Massachusetts

13. ABSTRACT

The work performed from 15 January to 15 April 1967 on Contract NAS12-59 is described in detail.

The report deals with the application of optimum linear estimation and control theory in designing an optical tracking system for use in deep-space communication. The estimator is obtained by using the extended Kalman filter, which necessitates the linearization of the equations of motion for the spacecraft and of the measurement equations. The controller is obtained by making the appropriate linearizations and applying new results in the theory of linear optimal control.

This study has resulted in the development of a digital computer program which simulates the operation of the estimator-controller in the optical tracking system.

DD FORM 1 NOV 65 1473 (PAGE 1)

S/N 0101-807-6801

UNCLASSIFIED

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Optical Tracking						
Optimum Estimation and Control Theory						
Kalman Filter						
Riccati Equation						